HAHN-BANACH THEOREM FOR NORMED SPACES

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ABSTRACT

This paper main objective is to present two separation theorems, important consequences of the Hahn-Banach Theorem applied to normed vector spaces. They are very important optimization tools in a lot of problems, namely in economic and management problems.

Keywords: Hahn-Banach Theorem, normed vector spaces, separation theorems.

1. INTRODUCTION

Definition 1.1

If *L* is a vector space, a norm in *L* is a functional *p* such that:

- $p(x) \geq 0,$
- $p(x) = 0, if\ and\ only\ if\ x = 0,$
- $p(x + y) \leq p(x) + p(y),$
- $p(\alpha x) = |\alpha| p(x), for\ any\ \alpha.$ ■

A vector space *L* with a norm is a normed space. It is usual to designate the norm of an element $x \in L$ for $\|x\|$.

Any normed space is a metric space which distance is given by

$$d(x, y) = \|x - y\|.$$

2. CONTINUOUS LINEAR FUNCTIONALS

Be *E* a normed vector space.

Definition 2.1

A linear functional *f*, defined in *E*, is continuous in $x_0 \in E$ *if and only if*, for any $\varepsilon > 0$, there is a neighbourhood *U* of x_0 such that

$$|f(x) - f(x_0)| < \varepsilon\ for\ x \in U.$$ ■

Definition 2.2

A linear functional *f*, defined in *E*, is continuous if it is continuous for all $x_0 \in E$. ■

Some important results on the continuity of linear functionals defined in normed vector spaces follow.

Proposition 2.1

Be *E* a normed vector space and *f* a linear functional in *E*. So

- If *E* is of finite dimension, *f* is continuous,
- *f* is continuous if and only if *f* is continuous at the origin,

- *f* is continuous if and only if *f* is bounded on the unitary ball.■

Definition 2.3

Be *f* a continuous linear functional in a normed space *E*. It is called *f* norm, $||f||$, to

$$||f|| = \sup_{||x|| \leq 1} |f(x)|$$

that is: to the supreme of the values that $|f(x)|$ assumes in the *E* unitary ball.■

Observation:

- The class of the continuous linear functionals, with the norm defined above, is a normed vector space, called the *E* dual space, designated E'.

3. THE HAHN-BANACH THEOREM FOR NORMED SPACES

Theorem 3.1 (Hahn-Banach)

Be *p* a positively homogeneous convex functional defined in a real vector space *L* and L_0 a *L* subspace. If f_0 is a linear functional defined in L_0, fulfilling the condition

$$f_0(x) \leq p(x), \forall_{x \in L_0}$$

so exists an extension *f* of f_0 defined in *L*, linear, and such that $f(x) \leq p(x), \forall_{x \in L}$. ■

For vector normed spaces theorem 3.1 assumes the form

Theorem 3.2 (Hahn-Banach)

Be *L* a subspace of a normed real space *E* and f_0 a linear functional bounded in *L*. So, there is a linear functional *f* defined in *E*, extension of f_0, such that

$$||f_0||_{L'} = ||f||_{E'}$$

Demonstration:

It is enough to think in the functional $k||x||$ such that $k = ||f_0||_{L'}$. As it is convex and positively homogeneous, it is possible to put $p(x) = k||x||$ and to apply theorem 3.1.■

The version of theorem 3.2 for complex spaces may be stated as

Theorem 3.3 (Hahn-Banach)

Be *E* a complex normed space and f_0 a bounded linear functional defined in a subspace $L \subset E$. So, there is a bounded linear functional *f,* defined in *E* such that

$$f(x) = f_0(x), x \in L, ||f||_{E'} = ||f_0||_{L'}. ■$$

A geometric interpretation of theorem 3.2 in normed spaces is as follows:

- Consider the equation $f_0(x) = 1$. It defines, in *L*, an hyperplane at distance $\frac{1}{\|f_0\|}$ from 0. Considering the extension of *f* to f_0, performed with norm conservation, it is obtained an hyperplane in *E*, that contains the hyperplane former considered in *L*. And it is at the same distance from the origin.

4. SEPARATION THEOREMS

The theorems presented in this section are important consequences of the Hahn-Banach theorem applied to normed vector spaces.

Definition 4.1

A convex field, in a vector space, is a convex set with non-empty null space.■

It may be shown that:

- In a normed space, the null space of a set matches with its interior.

So,

- In a normed space, a convex field is a convex set that contains at least an interior point.

Theorem 4.1 (Separation)

Be *A* and *B* two convex sets in a normed space *E*. If one of them, for instance *A*, has at least an interior point and $(int\,A) \cap B \equiv \emptyset$, there is a continuous linear functional non-null that separates the sets *A* and *B*.

Demonstration:

It is guaranteed the existence of a non-null functional that separates *A* and *B*. What is new, now, is that such functional is necessarily continuous. Note that if $\sup_{x\in A} f(x) \le \inf_{x\in B} f(x)$, f is superiorly bounded in *A*.

Being so, *f* is superiorly bounded in a ball $U(x_0)$, contained in *A*, centered in a point x_0 belonging to the interior of *A*. So it is also bounded inferiorly in $U(x_0)$. Then *f* is continuous because any linear functional bounded in a ball is continuous.■

Theorem 4.2 (Separation)

Given a closed convex set *A*, in a normed space *E*, and a point $x_0 \in E$, not belonging to *A*, there is a non-null continuous linear functional that separates strictly $\{x_0\}$ and *A*.

Demonstration:

Consider any convex neighbourhood of x_0, *U*, such that $U \cap A = \emptyset$. So the conditions of theorem 4.1 are fulfilled, and it may be concluded that there is a non-null continuous linear functional *f*, such that

$$\sup_{x\in U} f(x) \le \inf_{x\in A} f(x).$$

But, being x_0 an interior point of *U*, $x_0 + az \in U, \|z\| \le 1$, for any $a > 0$. Then, $\sup_{\|z\|\le 1} f(x_0 + az) \le \inf_{x\in A} f(x)$ and $f(x_0) + a \sup_{\|z\|\le 1} f(z) \le \inf_{x\in A} f(x)$, concluding that $f(x_0) < \inf_{x\in A} f(x)$. So *f* separates strictly $\{x_0\}$ and *A*.■

5. CONCLUSIONS

The fruitfulness of Hahn-Banach theorem is quite exemplified in this paper, through the resulting separation theorems, important to understand the functionals optimization problems.

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