OPTIMAL CONTROL THEORY AND IT'S APPLICATIONS IN AEROSPACE ENGINEERING

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ABSTRACT

Control theory is one of the most important mathematical milestones in present century. At present there are many branches of science and technology in which control theory plays a central role and faces fascinating challenges. In some cases, one expects to solve the problem by means of technologies development that will make possible to implement more sophisticated control mechanisms. In this study we have briefly mentioned some of the fields in which these challenges are present. Our main objectives are to investigate some aspects of calculus of variations and the control theory as well as their scopes and applications. We have studied here the applications of control theory in landing a space vehicle for optimal controlling the fuel.

Key words: Control theory, Space vehicle, Hamiltonian H.

1. INTRODUCTION

The Control theory is used in almost every fields of the modern sciences. The optimal control theory is a branch of dynamic optimization as well as it is a generalization of calculus of variations. See [1], [4], [5], [6], [11] and [14] for more details as well as for the history of control theory. The theory of optimization continues to be an area of active research not only for mathematicians but also for engineers and thus is an indication both of the inherent beauty of the subject and of its relevance to modern developments in engineering, science, industry and commerce. The optimal control is now playing a central role in many engineering applications, specially in the systems and control engineering such as robotics and aeronautics (see for examples [10] and [13]); in the life sciences such as sustainable forest management (see for example [2]); in the mathematical biology and medicine such as modeling and optimal controlling the infectious diseases (see for examples [8] and [9]). In the past few decades, there has been an overwhelming demand for the development of the technology enabling successful applications of control theory in aerospace engineering. In this paper our aim is to investigate the optimal control theory and some of its applications specially in controlling the fuel while landing of a space vehicle.

1.1 Determination of Hamiltonian H.

Hamiltonian plays a significant role in deriving the necessary conditions of optimality for optimal control problems (see [3] , [7] and [12] for details study on Hamiltonian). Before going to discuss our main issue, we first determine the Hamiltonian and in this case we shall restrict ourselves to one control variable u so that U is closed interval on the real line. The state equations are then

$$h' = u_1(x_1, x_2, t),\ m' = u_2(x_1, x_2, t)$$

Let $u^*(t)$ be an optimal control and $x^*(t)$ the corresponding optimal path. Consider a small variation of u^* such that $u = u^* + \partial u(t)$ with corresponding path $\left(x_1^* + \partial x_1, x_2^* + \partial x_2\right)$. This will not arrive at x_1 at t_1 but at a slightly different time $t_1 + \partial t$. The end conditions give $x_i^*(t_1+\partial t)+\partial x_i(t_1+\partial t)=x_i'$, $i=1,2$

As usual in variation arguments we are in the first instance interested only in first order effects and from the conditions we deduce that $\partial x_i(t_1)+x_i'^*(t_1)\partial t=0$, $i=1,2$

If we now use the state equations we obtain

$\partial x_i(t_1) = -u_i(t_1)\partial t$, where $u_i(t_1)$ denotes $u_i\left(x_1^*(t_1), x_2^*(t_1), u^*(t_1)\right)$.

To simplify the notation we will let u_i denoted by $u_i\left(x_1^*(t_1), x_2^*(t_1), u^*(t_1)\right)$ and we adopt the same convention for $\frac{\partial u_i}{\partial x_j}$ and $\frac{\partial u_i}{\partial u}$.

Then the consequent change ΔJ in J is

$$\Delta J = \int_{t_0}^{t_1+\partial t} u_0\left(x_1^* + \partial x_1, x_2^* + \partial x_2, u^* + \partial u\right) dt - \int_{t_0}^{t_1} u_0\left(x_1^*, x_2^*, u^*\right) dt$$

$$=\int_{t_0}^{t_1}\left\{\frac{\partial u_0}{\partial x_1}\partial x_1+\frac{\partial u_0}{\partial x_2}\partial x_2+\frac{\partial u_0}{\partial u}\partial u\right\}dt+u_0\left(t_1\right)\partial t+0\left(\left(\partial u\right)^2\right)$$

The derivatives in the integrand are evaluated on the optimal trajectory. Let ∂J denotes the first variation. If u^* is optimal it is necessary that the first variation ∂J is zero. So

$$\partial J=\int_{t_0}^{t_1}\left\{\frac{\partial u_0}{\partial x_1}\partial x_1+\frac{\partial u_0}{\partial x_2}\partial x_2+\frac{\partial u_0}{\partial u}\partial u\right\}dt+u_0(t_1)\partial t=0$$

on an optimal path for variations.

The partial derivatives ∂u ,$\partial x_1,\partial x_2$ are not independent here, they are linked by the state equations. These are because of the constrained optimal control problems dealt in many literatures (see for examples [4], [5], [11] and [14]) . In this case, we simply need to introduce two Lagrange multipliers $\phi_1\left(t\right)\text{and }\phi_2\left(t\right)$. We have chosen them to be time-dependent. Now consider pair of integrals

$$\phi_i=\int_{t_0}^{t_1}\varphi_1(t)(x_1'-u_i(x_1,x_2))dt\,,\quad i=1,2$$

They are both zero because the state equations must be satisfied. If we now let u^* be optimal and we calculate the first variation

$\partial\phi_i=0$ Since $\phi_i=0$ for all.

Then a straight forward calculation is given by

$$\partial\phi_i=\int_{t_0}^{t_1}\varphi_i(t)\left\{-\frac{\partial u_i}{\partial x_1}\partial x_1-\frac{\partial u_i}{\partial x_2}\partial x_2-\frac{\partial u_i}{\partial u}\partial u+\frac{d}{dt}(\partial x_i)\right\}dt$$

Now $\int_{t_0}^{t_1}\varphi_i(t)\frac{d}{dt}(\partial x_i)dt=\left[\varphi_i(t)\partial x_i\right]_{t_0}^{t_1}-\int_{t_0}^{t_1}\varphi_i'\partial x_i dt$

$$=-u_i(t_1)\varphi_i(t_1)\partial t-\int_{t_0}^{t_1}\varphi_i'\partial x_i dt$$

Since $\partial x_i(t_0)=0$ and $\partial x_i(t_1)=-u_i(t)dt$

Thus,

$$\partial\phi_i=-\int_{t_0}^{t_1}\varphi_i\left(t\right)\left\{\frac{\partial u_i}{\partial x_1}\partial x_1+\frac{\partial u_i}{\partial x_2}\partial x_2+\frac{\partial u_i}{\partial u}\partial u\right\}dt$$

$$-\int_{t_0}^{t_1}\varphi_i'\partial x_i dt-u_i\left(t_1\right)\varphi_i\left(t_1\right)\partial t=0$$

The condition that $\partial J=0$ can now be replaced by the condition that

$$\partial J+\partial\phi_1+\partial\phi_2=0$$

On substituting for $\partial J,\partial\phi_1\text{ and }\partial\phi_2$ and rearranging the terms we obtain,

$$\int_{t_0}^{t_1}\partial x_1\left\{\frac{\partial u_0}{\partial x_1}-\varphi_1\frac{\partial u_1}{\partial x_1}-\varphi_2\frac{\partial u_2}{\partial x_1}-\varphi_1'\right\}+$$

$$\int_{t_0}^{t_1}\partial x_2\left\{\frac{\partial u_0}{\partial x_2}-\varphi_1\frac{\partial u_1}{\partial x_2}-\varphi_2\frac{\partial u_2}{\partial x_2}-\varphi_2'\right\}dt+$$

$$\int_{t_0}^{t_1}\partial u\left\{\frac{\partial u_0}{\partial u}-\varphi_1\frac{\partial u_1}{\partial u}-\varphi_2\frac{\partial u_2}{\partial u}\right\}dt+$$

$$\left\{u_0\left(t_1\right)-u_1\left(t_1\right)\varphi_1\left(t_1\right)-u_1\left(t_1\right)\varphi_1\left(t_1\right)\right\}\partial t=0$$

This will be written more compactly if we introduce the Hamiltonian function

$$H=-u_0\left(x_1,x_2,u\right)+\varphi_1u_1\left(x_1,x_2,u\right)+\varphi_2u_2\left(x_1,x_2,u\right)$$

Then we have

$$\int_{t_0}^{t_1}\partial x_1\left\{\frac{\partial H}{\partial x_1}+\varphi_1'\right\}dt+\int_{t_0}^{t_1}\partial x_2\left\{\frac{\partial H}{\partial x_2}+\varphi_2'\right\}dt+\int_{t_0}^{t_1}\partial u\frac{\partial H}{\partial u}dt+H\left(t_1\right)\partial t=0$$

for admissible variations $\partial u, \partial x_1, \partial x_2$ where as usual, the derivatives are evaluated on the optimal path.

The multipliers φ_1, φ_2 are at our disposal and if we choose them to satisfy the equations

$$\varphi_i' = -\frac{\partial H}{\partial x_i}, \quad i = 1,2$$

Then the condition no longer involves ∂x_1 and ∂x_2. It becomes

$$\int_{t_0}^{t_1} \frac{\partial H}{\partial u} \partial u dt + H(t_1)\partial t = 0$$ for allowed variations.

Now we consider the variations $u^* + \partial u$ for which $\partial t = 0$; that is the corresponding solutions for x arrive at x' at $t = t'$ then our condition becomes

$$\int_{t_0}^{t_1} \frac{\partial H}{\partial u} \partial u dt = 0$$ for all admissible ∂u.

From this we can deduce that $\frac{\partial H}{\partial u} = 0$ at every point on an optimal trajectory. Furthermore we observe that if we allow variations for which $\partial t \neq 0$. We must still have $\frac{\partial H}{\partial u} = 0$ at every point. So that it is also necessary that $H(t_1) = 0$ at the end-point of an optimal trajectory. Thus a necessary condition for optimality is that $\frac{\partial H}{\partial u} = 0$ at each point on the optimal path and $H = 0$ at $t = t_1$ on the optimal path, where $H = -u_0 + \varphi_1 u_1 + \varphi_2 u_2$

and the function ψ_i satisfies the equations $\dot{\psi}_i = -\frac{\partial H}{\partial x_i}$.

These equations are called the *co-state* equations and H is sometimes referred to as the *Hamiltonian*.

2. FUEL OPTIMAL LANDING OF THE SPACE VEHICLE

We are now in a position to discuss our main problem; that is, the soft and optimal fuel consumed landing of a space vehicle. We assume that a space vehicle on a vertical trajectory tries to land smoothly on the surface of a planet. We denote by $h(t)$, the height at time t so that $v(t) = h'(t)$ is the instance velocity of the space vehicle (see for details [5]).

Since combustible is being consumed the mass $m(t)$ of the vehicle non increasing function of t, if we call $u(t)$ the instantaneous upwards thrust. Newton's law gives

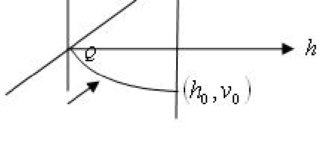
$m(t)h''(t) = -gm(t) + u(t)$ where g is the acceleration of gravity assuming that the thrust is proportional to the rate of decrease of mass is proportional to the rate at which combustible is used up $m'(t) \infty -u(t)$ which implies that $m'(t) = -ku(t)$, where $k > 0$

We introduce $v(t) = h'(t)$ as a variable and we obtain the following first order system of differential equation $h'(t) = v(t)$.

$$\text{Then, } mv'(t) = mh''(t) = -gm(t) + u(t) \quad \Rightarrow v'(t) = -g + \frac{u(t)}{m}$$

At the initial time $t_0 = 0$, we have initial conditions, $h(0) = h_0$, $v(o) = v_0$, $m(0) = m_0$.

The vehicle will land softly at time $\bar{t} \geq 0$ if $h(t) = 0$.



Trajectory of rocket in (h, v, m) plane

Trajectory of rocket in (h, v) plane

Fig. 1.(a) *Landing of a space vehicle*

The thrust cannot be negative or arbitrarily large $0 \le u(t) \le R$, for some $R > 0$.

We have an optimization problem if we try to land the space vehicle minimizing the amount of combustible

$$m(0) - m(\bar{t}) = k\int_0^{\bar{t}} u(t)dt = J(u)$$

$$H = \varphi_1 x_2 + \varphi_2 u_2 - u$$

is to be maximized as a function of u. Now we can write u in the form $|\, u \,| \operatorname{sgn} u$ and hence expression H in the form $S = \varphi_2 \operatorname{sgn} u - 1$.

There are three possibilities for $\operatorname{sgn} S$:

(i) If $|\, \varphi_2 \,|$<1 then $\operatorname{sgn} S < 0$, so H will be maximized by $u = 0$,

(ii) If $|\, \varphi_2 \,|$>1 then $\operatorname{sgn} S = \operatorname{sgn} \varphi_2 \operatorname{sgn} u$, so H will be maximized by $u = \operatorname{sgn} \varphi_2$

(iii) If $|\, \varphi_2 \,|$=1 then $S = \operatorname{sgn} \varphi_2 \operatorname{sgn} u - 1$ in which case

$$S = \begin{cases} -2 & \text{for } \operatorname{sgn} u = -\operatorname{sgn} \varphi_2 \\ 0 & \text{for } \operatorname{sgn} u = \operatorname{sgn} \varphi_2 \end{cases}$$

Thus we are forced to choose $\operatorname{sgn} u = \operatorname{sgn} \varphi_2$ and we find that the control is not completely determined; its sign is known but its magnitude is indeterminate. We can only say that $u = v(t) \operatorname{sgn} \varphi$ where $0 \le v(t) \le 1$.

Thus the control satisfying the Pontryagin maximum principle [10] can be written,

$$u^* = \begin{cases} 0 & \text{if } \psi_2 < 1 \\ \operatorname{sgn} \varphi_2 & \text{if } \psi_2 > 1 \\ v(t) \operatorname{sgn} \varphi_2 & \text{if } \psi_2 = 1 \\ 0 \le v(t) \le 1 & \end{cases}$$

The co-state variables are found to be $\varphi_1 = A, \varphi_2 = B - At$. $|\, \varphi_2 \,| = 1$ only at isolated times $(\text{since } A \ne 0)$ then u will be indeterminate at isolated instants as it switches between -1 and 0 or 1 and 0 . If $A = 0$ and $|\, B \,|$=1 then u^* is indeterminate for all t; the control is singular.

First let us consider the non-singular controls that maximize the Hamiltonian H.

Since $\psi_2 = B - At$ and $A \ne 0$, u^* can take only the values $+1, -1$ and 0. The corresponding trajectories are two families of parabolas $x_2^2 = 2u^* x_1 + k,\ u^* = \pm 1$ and a family of straight lines $x_2 = l$ corresponding to $u^* = 0$. Note that when $u^* = 0$ we have $x_1' = x_2$, $x_2' = 0$ so there is a line of singularities on $x_2 = 0$. This means that no optimal control can end with $u^* = 0$. Thus the only non-singular control sequences are

$$\{-1, 0, 1\},\ \{0, 1\},\ \{1\} \text{ and } \{1, 0, -1\},\ \{0, -1\},\ \{-1\} \tag{1}$$

since φ_2 is linear in t.

Unfortunately we cannot construct an optimal solution from a general initial point using these control sequences. Let us calculate the fuel consumed in going from an initial point (ξ_1, ξ_2) to $(0,0)$ using any admissible control.

On any path $x_2' = u, |\, u \,| \le 1$ so $x_2 = \xi_2 + \int_0^t u(t)dt$

Now $x_2(t_1) = 0$ so $0 = \xi_2 + \int_0^{t_1} u(t)dt$,

Hence $|\, \xi_2 \,| = |\, \int_0^{t_1} u(t)dt \,| \le \int_0^{t_1} |\, u(t) \,|\, dt = J$

and so $J \ge\ |\, \xi_2 \,|$.

If we find a $u(t)$ such that the corresponding value of J is $|\xi_2|$, then this must be optimal. We can show that there are some initial states for which there is an infinite number of fuel-optimal controls and other states for which there is no fuel-optimal control.

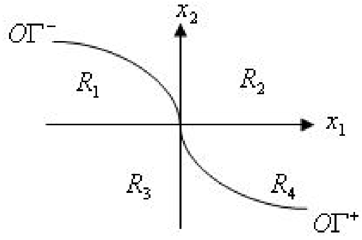


Fig. 1(b) *Phase Plane*

Let us divide the phase plane as shown in Fig. 1(b) and we observe that $O\Gamma^+$ is the half-parabola $x_2^2 = 2x_1, x_2 \le 0$ and similarly $O\Gamma^-$ is the half-parabola $x_2^2 = -2x_1, x_2 \ge 0.$ The region R_1 lies above $O\Gamma^-$ and $x_2 = 0, x_1 \ge 0$. It includes $x_2 = 0, x_1 \ge 0$ and excludes $O\Gamma^-$. The region R_2 is between $O\Gamma$ and $x_2 = 0, x_1 \ge 0$. It includes $O\Gamma^-$ and excludes $x_2 = 0, x_1 \le 0$.

Now it is easy to show that

(a) For (ξ_1, ξ_2) in R_1 or R_3 there is no optimal control.

(b) For (ξ_1, ξ_2) in R_2 or R_4 there are infinitely many optimal controls.

To prove (a) we consider (ξ_1, ξ_2) in R_3. Consider first the non-singular controls that maximize H. They are listed in (1) to get to O we need the control sequence $\{1, 0, -1\}$. The switch from $u^* = 1$ or $u^* = 0$ must take place at a point lying in R_2 with $x_2 = \varepsilon > 0$ and since $u^* = 1$ at the start we have $x' = 1$, so the time taken to get from $x_2 = \xi_2$ to $x_2 = \varepsilon$ is $t_1 = |\xi_2| + \varepsilon$. The control is then switched to $u^* = 0$ and the truck drifts uncontrolled (consuming no fuel) along $x_2 = \varepsilon$ until $O\Gamma^-$ is reached. Then u^* is switched to -1 and $t = \tau_2$ say, and the system gets to o with $x_2' = -1$ so $t_1 - t_2 = 0$

The fuel consumed is $J = \int_0^{\tau_1} |1|\, dt + \int_{\tau_1}^{\tau_2} 0 dt + \int_0^{t_1} |1\, dt|\, \mathrm{dt} = |\xi_2| + 2\varepsilon$.

No control sequence $\{1, 0, -1\}$ can give J its known minimum value. Singular controls arise when $\phi_2 = 1 \text{ or } -1$, not just an isolated instant but for a time interval. This will happen if $A = 0, B = 1$ or -1. Such controls, which are of the form $v(t)\operatorname{sgn} B$, $0 \le v(t) \le 1$, cannot change the sign. This means that no initial state in R_3 can be driven to the origin by the singular control that maximizes H. To see this, note that in R_3 the x_2-coordinate is negative that to drive the system closer to the origin we need $u(0) > 0$. Since the singular control cannot change sign, x_2 increases for all t and the system is driven infinity. It is impossible to reach O from R_3 using a singular control. Thus there is no optimal control for (ξ_1, ξ_2) in R_3.

To prove (b) we consider an initial state in R_4. Suppose the control is non-singular. If we take $u^* = 0$ until the system has drifted along $x_2 = \xi_2$ to a point on $O\Gamma^+$ and then switch to $u = 1$, we can control the system to the origin and the value of J is $|\xi_2|$. This is an optimal control. Now suppose the control is singular. We must have $u = v(t),\ 0 \le v(t) \le 1$ with corresponding state equations $x_1' = x_2$, $x'_2 = v(t)$

which is integrated to give

$$x_2 = \xi_2 + \int_0^t v(\sigma) d\sigma,$$

$$x_1 = \xi_1 + \int_0^t \left\{ \xi_2 + \int_0^\tau v(\sigma)d\sigma \right\} d\tau$$

At time t_1 the system is to be at $x_1 = x_2 = 0$, so

$$0 = \xi_2 + \int_0^{t_1} v(\sigma)d\sigma$$

$$= \xi_1 + \int_0^{t_1} \xi_2 d\tau + \int_0^{t_1} \int_0^{\tau} v(\sigma)d\sigma d\tau \Rightarrow -\xi_2 = \int_0^{t_1} v(\sigma)d\sigma$$

$$\Rightarrow -\xi_1 - \xi_2 t_1 = \int_0^{t_1}\int_0^{\tau} v(\sigma)d\sigma d\tau \tag{2}$$

Thus there are an infinite number of functions $v(\sigma),\ 0 \le v(\sigma) \le 1$ that satisfy (2). They are all optimal since $J = \int_0^{t_1} | u | \, dt = \int_0^{t_1} v(\sigma)d\sigma = | \xi_2 |$

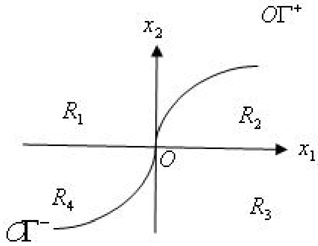


Fig. 1 (c). *Phase Plane*

Again, to prove (a) let us consider (ξ_1, ξ_2) in R_1. Consider first the non-singular controls that maximize H. They are listed in (1) to get to O we need the control sequence $\{1, 0, -1\}$. The switch from $u^* = 1$ or $u^* = 0$ must take place at a point lying in R_4 with $x_2 = \varepsilon > 0$ and since $u^* = 1$ at the start we have $x' = 1$, so the time taken to get from $x_2 = \xi_2$ to $x_2 = \varepsilon$ is $t_1 = | \xi_2 | + \varepsilon$. The control is then switched to $u^* = 0$ and the truck drifts uncontrolled (consuming no fuel) along $x_2 = \varepsilon$ until $O\Gamma^-$ is reached. Then u^* is switched to -1 and $t = \tau_2$ say, and the system gets to O with $x_2' = -1$ so $t_1 - t_2 = 0$

The fuel consumed is

$$J = \int_0^{\tau_1} |1| \, dt + \int_{\tau_1}^{\tau_2} 0 dt + \int_0^{t_1} | 1 \, dt | \text{dt} = | \xi_2 | + 2\varepsilon .$$

No control sequence $\{1, 0, -1\}$ can give J its known minimum value. Singular controls arise when $\phi_2 = 1 \text{ or } -1$, not just an isolated instant but for a time interval.

This will happen if $A = 0, B = 1$ or -1. Such controls, which are of the form $v(t)\operatorname{sgn} B$, $0 \le v(t) \le 1$, cannot change the sign. This means that no initial state in R_1 can be driven to the origin by the singular control that maximizes H. To see this, note that in R_1 the x_2-coordinate is negative that to drive the system closer to the origin we need $u(0) > 0$. Since the singular control cannot change sign, x_2 increases for all t and the system is driven infinity. It is impossible to reach O from R_1 using a singular control.

Also to prove (b) we consider an initial state in R_2. Suppose the control is non-singular. If we take $u^* = 0$ until the system has drifted along $x_2 = \xi_2$ to a point on $O\Gamma^+$ and then switch to $u = 1$, we can control the system to the origin and the value of J is $|\xi_2|$. This is an optimal control. Now suppose the control is singular. We must have $u = v(t)$, $0 \le v(t) \le 1$ with corresponding state equations $x_1' = x_2$, $x'_2 = v(t)$

which is again integrated to give $x_2 = \xi_2 + \int_0^t v(\sigma)d\sigma$, $x_1 = \xi_1 + \int_0^t \left\{ \xi_2 + \int_0^\tau v(\sigma)d\sigma \right\} d\tau$

At time t_1 the system is to be at $x_1 = x_2 = 0$, so

$$0 = \xi_2 + \int_0^{t_1} v(\sigma)d\sigma = \xi_1 + \int_0^{t_1} \xi_2 d\tau + \int_0^{t_1} \int_0^{\tau} v(\sigma)d\sigma d\tau$$

$$\Rightarrow -\xi_2 = \int_0^{t_1} v(\sigma)d\sigma$$

$$\Rightarrow -\xi_1 - \xi_2 t_1 = \int_0^{t_1}\int_0^{\tau} v(\sigma)d\sigma d\tau$$

Thus there are an infinite number of functions $v(\sigma)$, $0 \le v(\sigma) \le 1$ that satisfy (2). They are all optimal since $J = \int_0^{t_1} |u| \, dt = \int_0^{t_1} v(\sigma)d\sigma = |\xi_2|$.

3. APPLICATIONS OF CONTROL THEORY

In this section, we will discuss some applications of control theory. We will illustrate two examples in this regard. The Pontryagin maximum principle [12] is a useful necessary condition which we can now use to solve a range of control problems. We look first at the problem of controlling a linear system in a time optimal manner. The truck problem is the simplest two-dimensional problem of this type. The general problem is dealt with in the next section. For the truck problem the control $u = u(t)$ was subject to the constraint $|u| \le \frac{k}{m}$; in what follows the constraint has been normalized to $|u| \le 1$ but there is no loss of generality. As was explained earlier we can also set $\psi_0 = -1$ in any application of the Pontryagin theorem without loss of generality.

Problem 1.

Suppose the system $\dot{x}_1 = f_1(x_1, x_2, u)$, $\dot{x}_2 = f_2(x_1, x_2, u)$ is to be controlled from x_0 at t_0 to some point on the curve $g(x_1, x_2) = 0$ at some time t_1 in such a way that $J = \int_{t_0}^{t_1} f_0(x_1, x_2, u)dt$ is minimized. Find the optimal control.

Solution: Suppose that the problem has been solved so that u^* controls the system from x_0 to a point on the target curve l' and minimizes J. In the augmented state space the optimal path ends at the point D on the curve l' defined by $g(x_1, x_2) = 0$, $x_0 = x_0^*$ where x_0^* is the minimum value of the cost J

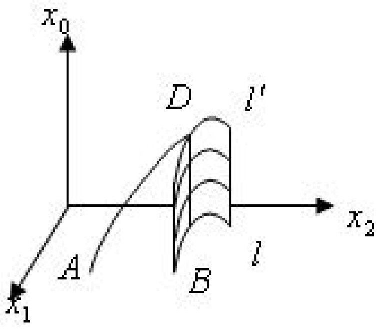


Fig. 2(a). *The Optimal Path*

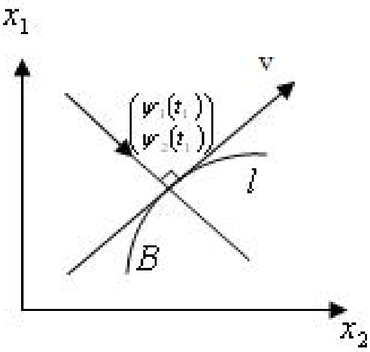


Fig 2(b). *The Optimal Path*

Since u^* minimizes J, so the set of varied end-points E must have a plane of support Π at D. Recall that Π is such that E lies on side of Π and the half-line ℓ on the other. The tangent to ℓ at D either lies entirely in Π or passes through Π at D. We shall show that the second possibility leads to a contradiction, so for optimality the tangent to ℓ' at D must lie in Π. This geometric result can be expressed as a simple condition involving the co-state variables at t_1 and the tangent to the target curve in state space; the two-dimensional vector $(\psi_1(t_1),\psi_2(t_1))^T$ must be perpendicular to the tangent to the target ℓ at the optimal end-point. This is the transversality condition. We can write it as follows: let $v=(\lambda,\mu)^T$ be the tangent to ℓ at $\left(x_1^*(t_1),x_2^*(t_1)\right)$ then $(\psi_1(t_1)\lambda+\psi_2(t_1)\mu)=0$ at the end-point.

3.1 Time-optimal Control of Linear Systems

We consider here systems with two variables $x_1(t), x_2(t)$ describing the state of the system and a single control variable $u(t)$ that is forced to take its values in such a way that $|u|\le 1$. We allow *u* to be piecewise continuous and let the system be governed by a pair of linear differential equations,

$$\dot{x} = ax_1 + bx_2 + lu \qquad \dot{x}_2 = cx_1 + dx_2 + mu$$

with $|u|\le 1$ and a,b,c,d,l,m are given constants.

In matrix notation the above system can be written as $\dot{x} = Ax + lu$

$$\text{where } A=\begin{pmatrix} a & b \\ c & d\end{pmatrix} \text{ and } l=\begin{pmatrix} l \\ m\end{pmatrix}$$

Problem 2: Given that system $\dot{x} = Ax + lu$ can be controlled from a given initial point $x(t_0)=x^0$ to a given target point $x(t_1)=x^1$ by an admissible control (piecewise continuous and taking its values from the set *U* such that $|u|\le 1$), find the optimal control $u^*(t)$ for which $J=\int_{t_0}^{t_1} 1dt = t_1 - t_0$ is minimized .

Solution: We first need to write down the Hamiltonian $H(\Psi,x,u)$ and find the co-state equations and then maximize H as function of Ψ

We know that the Hamiltonian is given as,

$$H(\Psi,x,u) = -1+\Psi_1(ax_1+bx_2+lu)+\Psi_2(cx_1+dx_2+mu)$$

$$=-1+\Psi_1(ax_1+bx_2+)+\Psi_2(cx_1+dx_2+)+(l\Psi_1+m\Psi_2)u$$

Now the co- state equations are derived as

$$\Psi_1 = -\frac{\partial H}{\partial x_1} = -a\Psi_1 - c\Psi_2\text{, } \Psi_2 = -\frac{\partial H}{\partial x_2} = -b\Psi_1 - d\Psi_2$$

Or in matrix notation, we have $\dot{\Psi} = -A^t\Psi$ where $\Psi = \begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix}$.

We now choose at each value of t the value of $u=u(t)$ that maximizes the Hamiltonian. We note that, H is linear in u, so to maximizes H we need $u=1$ or $u=-1$, depending on the sign of the coefficient $l\Psi_1+m\Psi_2$.

Thus the only controls that can lead to a minimum time of transfer are those of the form,

$u^* = \mathrm{Sgn}(l\Psi_1+m\Psi_2)$.

They are piecewise constant controls that are discontinuous at the zeros of

$$S = l\Psi_1(t)+m\Psi_2(t). \tag{3}$$

That is, they switch from 1 to -1 or from -1 to 1 whenever $S=0$. For this reason S as defined by (3) is called the *switching function*. In the interval between two zeros of S the control is constant, so the state equations become autonomous,

$\dot{x} = Ax + 1u^*$, where $u^* = 1$ or -1.

And the form of the trajectories in the x_1x_2 - plane is easily found in each case. Provided that $ad-bc\ne 0$

The trajectories for $u^*=1$ will have an isolated singularity at the intersection of

$ax_1+bx_2+l=0$ and $cx_1+dx_2+m=0$

While the trajectories for $u^* = -1$ will have an isolated singularity at the intersection of

$ax_1 + bx_2 - l = 0$ and $cx_1 + dx_2 - m = 0$

The behavior of both families of trajectories is determined by the eigenvalues of the system matrix A . The trajectory pattern (their shape and the direction in which they are swept out as t increases) is the same as the pattern of the trajectories of the uncontrolled autonomous system $\dot{x} = Ax$.

The only difference is that the whole phase plane pattern is translated so that the singularity is at the solution of $ax_1 + bx_2 + l = 0$ and $cx_1 + dx_2 + m = 0$ for $u^* = 1$ and the solution of $ax_1 + bx_2 - l = 0$ and $cx_1 + dx_2 - m = 0$ for $u^* = -1$.

We recall that a singular point in the phase-plane represents a solution that is constant for all *t*. None of the trajectories of the system can pass through (or begin or end at) a singular point. In the following examples we use phrases such as 'the path *RO*'. Occasionally the point *R* is singularity and, strictly speaking, we should say 'the path *RO* with the singularity at *R* excluded' but doing so would be very cumbersome. Provided the reader bears this in mind there should be no confusion.

4. CONCLUSION

Now-a-days the applications of control theory to the real world problems have been more crucial than the theoretical aspects. So bridging between theory and real world applications is the main objective to the present day research in control theory. In this study the application of control theory in the aerospace dynamics is investigated. We discussed the application of control theory and represented some problems on that theorem. We have also discussed the time optimal control of linear systems. We have applied the theorem on landing a space vehicle for optimal controlling its fuel. Finally we conclude from our discussion in Section 2 that in the regions R_1 and R_3 , it is impossible to land a space vehicle controlling its fuel. But only in the region R_2 and R_4 , the space vehicle can be landed with optimal controlling its fuel.

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