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# THE TIES BETWEEN THE $M|G|\infty$ QUEUE SYSTEM TRANSIENT BEHAVIOUR AND THE BUSY PERIOD

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**ABSTRACT**

This paper main subject is the $M|G|\infty$ queue system transient probabilities study as time functions. It is completely achieved when the time origin is an unoccupied system instant. But such a goal is not got when the time origin is a busy period beginning instant. In this last situation, the service time length distribution hazard rate function plays a very important role. And so the results obtained may be useful in the survival analysis field. As the $M|G|\infty$ queue system can be applied to model many social problems: sickness, unemployment, emigration, ...(see, for instance, 1-2), in these situations it is very important to study the busy period length distribution of that system. Along this work it will be evidenced that the solution of the problem may be in the resolution of a Ricatti equation generalizing the work of Ferreira (3) as a consequence of the transient behavior study.

***Key words:*** $M|G|\infty$, transient behavior, hazard rate function, busy period, Ricatti equation

## 1. INTRODUCTION

$M|G|\infty$ is a queue system at which the customers arrive according to a Poisson process at rate $\lambda$, receive a service which time length is a positive random variable with distribution function $G(.)$ and mean $\alpha$. So $\alpha = \int_0^\infty [1-G(v)]dv$. Upon its arrival each customer finds immediately an available server. Each customer service is independent from the other customers services and from the arrival process. The traffic intensity is $\rho = \lambda\alpha$.

Being $N(t)$ the occupied servers number at time $t$ in a $M|G|\infty$ system, that for this queue system is the same that the being served customers number, according to Carrillo (4), putting $p_{0n}(t) = P[N(t)=n|N(0)=0]$, $n = 0,1,2,...$,

$$p_{0n}(t) = \frac{\left(\lambda\int_0^t [1-G(v)]dv\right)^n}{n!} e^{-\lambda\int_0^t [1-G(v)]dv}, n = 0,1,2,... \qquad (1.1).$$

So, the transient distribution, being the time origin an empty system instant, is Poisson with mean $\lambda\int_0^t [1-G(v)]dv$.

The stationary distribution is the limit one, Poisson with mean $\rho$:

$$\lim_{t\to\infty} p_{0n}(t) = \frac{\rho^n}{n!} e^{-\rho}, n = 0,1,2,...$$

Be $p_{1'n}(t) = P[N(t)=n|N(0)=1'], n = 0,1,2,...,$ meaning $N(0)=1'$ that the time origin is the one of a customer arrival at the system, making the number of served customers jump from $0$ to $1$. That is: a busy period begins.

At $t \geq 0$, possibly:

- The customer that arrived at the time origin abandoned the system, with probability $G(t)$, or goes on being served, with probability $1-G(t)$;
- The other servers, that were unoccupied at the time origin, are still unoccupied or occupied with $1,2,...$ customers, with probabilities given by $p_{0n}(t), n=1,2,...$

Both subsystems, the one of the initial customer and the other of the initially unoccupied servers, are independent and so

$$p_{1'0}(t) = p_{00}(t)G(t) \qquad (1.2)$$

$$p_{1'n}(t) = p_{0n}(t)G(t) + p_{0n-1}(t)(1-G(t)), n=1,2,...$$

For the $M|M|\infty$ system (exponential service times) (1.2) is valid even if $N(0)=1$, that is: since the time origin in an instant at which there is one only customer in the system, simply, owing to the exponential distribution lack of memory. So

$$p_{10}^{M}(t) = \left(1-e^{-\frac{t}{\alpha}}\right)e^{-\rho\left(1-e^{-\frac{t}{\alpha}}\right)}$$

$$p_{1n}^{M}(t) = \frac{1}{(n-1)!}\rho^{n-1}\left(1-e^{-\frac{t}{\alpha}}\right)^{n-1}e^{-\rho\left(1-e^{-t/\alpha}\right)}\left(\frac{\left(1-e^{-\frac{t}{\alpha}}\right)^{2}\rho}{n}+e^{-\frac{t}{\alpha}}\right), n=1,2,...$$

It is easy to check that

$$\lim_{t\to\infty} p_{1'n}(t) = \frac{\rho^n}{n!}e^{-\rho}, n=0,1,2,...$$

Calling $\mu(1',t)$ and $\mu(0,t)$ the mean values associated to the distributions given by (1.2) and (1.1), respectively,

$$\mu(1',t) = \sum_{n=1}^{\infty} np_{1'n}(t) = \sum_{n=1}^{\infty} nG(t)p_{00}(t) + \sum_{n=1}^{\infty} np_{0n-1}(t)(1-G(t)) =$$

$$G(t)\mu(0,t) + (1-G(t))\sum_{j=0}^{\infty}(j+1)p_{0j}(t) = (0,t)+(1-G(t)).$$ So

$$\mu(1',t) = 1-G(t)+\lambda\int_0^t[1-G(v)]dv \qquad (1.3).$$

Now some results about $p_{0n}(t), n = 0,1,2,...,$ $p_{1'0}(t)$ and $\mu(1',t)$ behaviours as time functions will be presented. And it will be evidenced also that the $p_{1'0}(t)$ study induces a Ricatti equation important to the determination of a $M \mid G \mid \infty$ systems collection with practically exponential busy period.

## 2. $p_{0n}(t)$, $n = 0,1,2,...$ BEHAVIOUR AS TIME FUNCTION

This section main result is:

**Proposition 2.1**

If $G(t) < 1$, $t > 0$, continuous and differentiable

a) $p_{00}(t)$, $t > 0$ is a decreasing function,

b) $p_{0n}(t)$, $n \geq \rho$, $t > 0$ is an increasing function,

c) $p_{0n}(t)$, $0 < n < \rho$, $\rho > 1$

**i) increases in $]0, t_n[$ being $t_n$ given by**

$$\int_0^{t_n} [1 - G(v)]dv = \frac{n}{\lambda} \qquad (2.1),$$

ii)decreases in $]t_n, \infty[$ and

iii) the $p_{0n}(t)$ maximum is

$$p_{0n}(t_n) = \frac{n^n}{n!} e^{-n} \qquad (2.2)$$

Dem: a) is evident since

$$p_{00}(t) = e^{-\lambda \int_0^t [1-G(v)]dv}.$$

For $n \geq 1$, $\frac{d}{dt} p_{0n}(t) = \lambda p_{0n}(t)(1 - G(t))\left(\frac{n}{\lambda \int_0^t [1 - G(v)]dv} - 1\right), t > 0$.

As $\lambda \int_0^t [1 - G(v)]dv < \rho$, if $n \geq \rho, \frac{d}{dt} p_{0n}(t) > 0, t > 0$.

And if $n < \rho, \frac{d}{dt} p_{0n}(t) = 0 \Leftrightarrow \int_0^t [1 - G(v)]dv = \frac{n}{\lambda}$.

**Notes:**

- Although $t_n$, given by (2.1), depends on the arrival rate and on the service time length distribution, that does not happen with $p_{0n}(t_n)$ given by (2.2),

- For a certain arrival rate and service time length distribution, evidently,

$$t_{n+1} \geq t_n$$

and, as

$$\frac{p_{0n+1}(t_{n+1})}{p_{0n}(t_n)} = \frac{(n+1)^{n+1}}{(n+1)!}e^{-n-1}\frac{n!}{n^n}e^n = (n+1)\left(\frac{n+1}{n}\right)^n \frac{e^{-1}}{n+1} =$$

$$= \left(1+\frac{1}{n}\right)e^{-1} \le ee^{-1} = 1,\ p_{0n+1}(t_{n+1}) \le p_{on}(t_n)$$

- Under Proposition (2.1) conditions, but with $1-G(t)=0, t \ge t_l$, (1.1) becomes

$$p_{0n}(t) = \frac{\left(\lambda\int_0^t [1-G(v)]dv\right)^n}{n!}e^{-\lambda\int_0^t [1-G(v)]dv}, t \le t_l \quad \text{and} \quad p_{0n}(t) = \frac{\rho^n}{n!}e^{-\rho}, t > t_l,$$

$n = 0,1,2$,... and, so, the Proposition (2.1) conclusions are still valid, but the values $\frac{\rho^n}{n!}e^{-\rho}, n = 0,1,2,...$ occur after $t = t_l$. Evidently, $t_n < t_l, 0 < n < \rho, \rho > 1$.

**3. $p_{1'0}(t)$ BEHAVIOUR AS TIME FUNCTION**

For the $p_{1'n}(t), n = 0,1,2,...$ it is not possible to perform such a complete study as for the $p_{0n}(t), n = 0,1,2,...$. But the results for $p_{1'0}(t)$ are very interesting. Now the important result is

**Preposition 3.1**

If $G(t) < 1, t > 0$, continuous, differentiable and

$$h(t) \ge \lambda G(t), t > 0 \qquad (3.1),$$

being $h(t)$ the hazard rate function associated to $G(.)$, $p_{1'0}(t)$ is non-decreasing.

**Dem:** It is enough to note that, under these conditions,

$$\frac{d}{dt}p_{1'0}(t) = p_{00}(t)(1-G(t))\left(\frac{g(t)}{1-G(t)} - \lambda G(t)\right),\ g(t) = \frac{d}{dt}G(t) \text{ and } h(t) = \frac{g(t)}{1-G(t)}.$$

**Notes:**

- Note that

$$h(t) \ge \lambda \qquad (3.2)$$

is a sufficient condition for (3.1) and so if the rate at which the services end is greater or equal than the customers arrival rate $p_{1'0}(t)$ is non-decreasing,

- For the $M|M|\infty$ system (3.2) is equivalent to

$$\rho \le 1$$

- Evidently these results may be useful in the survival analysis fields.

Putting

$h(t)-\lambda G(t)=\beta, \beta \in IR$

It is obtained a Ricatti equation which solution is (note that $G(t)=1,\ t\geq 0$ is a solution)

$$G(t)=1-\frac{\left(1-e^{-\rho}\right)(\lambda+\beta)}{\lambda e^{-\rho}\left(e^{(\lambda+\beta)t}-1\right)+\lambda}, t\geq 0, -\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1} \quad (3.3),$$

see Ferreira (3). For a $M|G|\infty$ system with this service time length distribution

$$p_{1'0}(t)=e^{-\rho}-\frac{\left(1-e^{-\rho}\right)\beta}{\lambda}e^{-(\lambda+\beta)t}, t\geq 0, -\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1}.$$

Concretely

- $\beta=-\lambda$

$p_{1'0}(t)=1, t\geq 0$.

In this situation, $G(t)=1, t\geq 0$. So $G(.)$ is degenerated at the origin. That is: every customer has null service time length. So the system is never occupied,

- $\beta=0$

$p_{1'0}(t)=e^{-\rho}, t\geq 0$

and so $p_{1'0}(t), t\geq 0$ is constant,

- $\beta=\dfrac{\lambda}{e^{\rho}-1}$

$p_{1'0}(t)=e^{-\rho}\left(1-e^{-(\lambda+\beta)t}\right), t\geq 0$

With the service time length distribution given by (3.3), (1.1) becomes

$$p_{0n}(t)=\frac{\left(-\log\left(e^{-\rho}+\left(1-e^{-\rho}\right)e^{-(\lambda+\beta)t}\right)\right)^{n}}{n!}\left(e^{-\rho}+\left(1-e^{-\rho}\right)e^{-(\lambda+\beta)t}\right),$$

$$, t\geq 0, -\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1}.$$

Calling $T$ the random variable associated to $G(.)$ given by (3.3) we have, see Ferreira (5)

$$\frac{\left(1-e^{-\rho}\right)e^{-\rho}}{\lambda}\frac{n!}{(\lambda+\beta)^{n-1}}\leq E\left[T^{n}\right]\leq\frac{e^{\rho}-1}{\lambda}\frac{n!}{(\lambda+\beta)^{n-1}},$$

$$,-\lambda < \beta \le \frac{\lambda}{e^{\rho}-1}, n = 1,2,... \quad (3.4)$$

**Notes:**

- The expression (3.4) giving bounds for $E\left[T^{n}\right], n = 1,2,...$ proves its existence,

- For $n = 1$ (3.4) is not useful because it is known that $E[T] = \alpha$ . Curiously, the upper bound is $\frac{e^{\rho}-1}{\lambda}$ , the $M|G|\infty$ system busy period mean value,

- For $\beta = -\lambda, E\left[T^{n}\right] = 0, n = 1,2,...$ evidently.

See however that (3.3) may take the form

$$G(t) = \frac{1+\frac{\beta}{\lambda}\left(1-e^{\rho}\right)e^{-(\lambda+\beta)t}}{1-\left(1-e^{\rho}\right)e^{-(\lambda+\beta)t}}, t \ge 0, -\lambda \le \beta \le \frac{\lambda}{e^{\rho}-1}$$

and, since $\rho < \log 2$ ,

$$G(t) = \left(1+\frac{\beta}{\lambda}\left(1-e^{\rho}\right)e^{-(\lambda+\beta)t}\right)\sum_{k=0}^{\infty}\left(1-e^{\rho}\right)^{k} e^{-k(\lambda+\beta)t},$$

$$, t \ge 0, -\lambda \le \beta \le \frac{\lambda}{e^{\rho}-1} \quad (3.5).$$

After (3.5) it is easy to compute the $T$ Laplace transform for $\rho < \log 2$ . And then obtain

$$E\left[T^{n}\right] = -\left(1+\frac{\beta}{\lambda}\right)n!\sum_{k=1}^{\infty}\frac{\left(1-e^{\rho}\right)^{k}}{\left(k(\lambda+\beta)\right)^{n}}, -\lambda < \beta \le \frac{\lambda}{e^{\rho}-1}, \rho < \log 2, n = 1,2,...$$

**Notes:**

- $E[T] = -\left(1+\frac{\beta}{\lambda}\right)\sum_{k=1}^{\infty}\frac{\left(1-e^{\rho}\right)^{k}}{k(\lambda+\beta)} = -\frac{\lambda+\beta}{\lambda(\lambda+\beta)}.\sum_{k=1}^{\infty}\frac{\left(1-e^{\rho}\right)}{k} =$

$$= \frac{1}{\lambda}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\left(e^{\rho}-1\right)^{k}}{k} = \frac{1}{\lambda}\log e^{\rho} = \frac{\rho}{\lambda} = \alpha$$

- For $n \geq 2$ it is imperious to truncate the infinite sum. Taking only $M$ terms, to achieve an error lesser or equal than $\varepsilon$ , M must simultaneously verify

. $M > \frac{1}{\lambda + \beta} - 1$

. $M > \log_{\left(e^{\rho} - 1\right)} \frac{e^{\rho}\lambda}{n!(\lambda + \beta)} - 1$

**4. A $M|G|\infty$ SYSTEMS COLLECTION WITH EXPONENTIAL BUSY PERIOD**

Define $h(t) - \lambda G(\lambda) = \beta(t)$ equivalent to

$\frac{dG(t)}{dt} = -\lambda G^2(t) - (\beta(t) - \lambda)G(t) + \beta(t)$ that is a Ricatti equation on $G(.)$.

Solving it, after noting that $G(t) = 1, t \geq 0$ is a solution again, it is obtained

$$G(t) = 1 - \frac{1}{\lambda} \frac{\left(1 - e^{-\rho}\right)e^{-\lambda t - \int_0^t \beta(u)du}}{\int_0^{\infty} e^{-\lambda w - \int_0^w \beta(u)du} dw - \left(1 - e^{-\rho}\right)\int_0^t e^{-\lambda w - \int_0^w \beta(u)du} dw},$$

$$, t \geq 0, -\lambda \leq \frac{\int_0^t \beta(u)du}{t} \leq \frac{\lambda}{e^{\rho} - 1} \qquad (4.1).$$

Substituting (4.1) in

$$\overline{B}(s) = 1 + \lambda^{-1}\left( s - \frac{1}{\int_0^{\infty} e^{-st - \lambda\int_0^t [1 - G(v)]dv} dt} \right)$$

the $M|G|\infty$ busy period length Laplace transform, see Stadje (6),

$$\overline{B}(s) = \frac{1 - (s + \lambda)(1 - G(0))L\left[e^{-\lambda t - \int_0^t \beta(u)du}\right]}{1 - \lambda(1 - G(0))L\left[e^{-\lambda t - \int_0^t \beta(u)du}\right]}, -\lambda \leq \frac{\int_0^t \beta(u)du}{t} \leq \frac{\lambda}{e^{\rho} - 1} \qquad (4.2)$$

where $L$ means Laplace transform and

$$G(0) = \frac{\lambda\int_0^{\infty} e^{-\lambda w - \int_0^w \beta(u)du} dw + e^{-\rho} - 1}{\lambda\int_0^{\infty} e^{-\lambda w - \int_0^w \beta(u)du} dw}.$$

After (4.2) it is possible to compute $\frac{1}{s}\overline{B}(s)$ which inversion is

$$B(t)=\left(1-(1-G(0))\left(e^{-\lambda t-\int_0^t \beta(u)du}+\lambda\int_0^t e^{-\lambda w-\int_0^w \beta(u)du}dw\right)\right)*$$

$$*\sum_{n=0}^{\infty}\lambda^n(1-G(0))^n\left(e^{-\lambda t-\int_0^t \beta(u)du}\right)^{*n},-\lambda\leq\frac{\int_0^t \beta(u)du}{t}\leq\frac{\lambda}{e^{\rho}-1} \qquad (4.3)$$

for the $M|G|\infty$ busy period d.f., where $*$ is the convolution operator.

If $\beta(t)=\beta$ (constant), we (4.3) is (3.3) and

$$B^{\beta}(t)=1-\frac{\lambda+\beta}{\lambda}\left(1-e^{-\rho}\right)e^{-e^{-\rho}(\lambda+\beta)t},t\geq 0,-\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1}.$$

So, if the service time d.f. is given by (3.3) the $M|G|\infty$ busy period d.f. is the a mixture of a degenerate distribution at the origin and an exponential distribution.

**Finally note that, for $\beta=\frac{\lambda}{e^{\rho}-1}$, $B^{\beta}(t)=1-e^{-\frac{\lambda}{e^{\rho}-1}t}$, $t\geq 0$ (purely exponential). And $B(t)$, given by (4.3) satisfies**

$$B(t)\geq 1-e^{-\frac{\lambda}{e^{\rho}-1}t},\ t\geq 0,\ -\lambda\leq\frac{\int_0^t \beta(u)du}{t}\leq\frac{\lambda}{e^{\rho}-1}.$$

## 5. $\mu(1't)$ BEHAVIOUR AS TIME FUNCTION

In this situation:

**Proposition 5.1**

If $G(t)<1, t>0$ , continuous, differentiable and

$$h(t)\leq\lambda \qquad (5.1)$$

$\mu(1',t)$ is non-decreasing.

Dem: After (1.3) we have $\frac{d}{dt}\mu(1',t)=(1-G(t))(\lambda-h(t))$.

**Notes:**

- If the rate at which the services end is lesser or equal than the customers arrival rate $\mu(1',t)$ is non-decreasing,

- For the *M*|*M*|∞ (5.1) is equivalent to $\rho \geq 1$

- These results may be useful in the survival analysis field.

If

$h(t) = \lambda$

obviously

$G(t) = 1 - e^{-\lambda t}, t \geq 0$

and, for this *M*|*M*|∞ system,

$\mu(1,t) = 1$

**6. CONCLUDING REMARKS**

In queues practical applications often it is used the population process stationary distribution. This happens generally because the transient distribution is very complex and not useful. And so the stationary distribution is used as a good transient one approximation. But in various situations this is not true. So it is necessary to know as well as possible the transient behaviour.

The *M*|*G*|∞ systems transient behaviour, with an unoccupied system instant time origin, is very well known and not too complex. It was deduced, in this work, for the time origin at the beginning of a busy period the transient distribution.

The transient behavior study is presented here, with some interesting results, for the M|G|∞ systems with a lot of possible applications, namely in survival analysis.

It was done more exhaustively for the $p_{0n}(t)$ than for the $p_{1'n}(t), n = 0,1,2,...$, but in the former situation everything is easier than in the other. But the $p_{1'0}(t)$ study leads to very interesting results even that they are looked only from the mathematical point of view. And, no less important, it allows through the resolution of a Ricatti equation the determination of a M|G|∞ infinite systems collection with a very simple busy period distribution: a mixture of a degenerate distribution at the origin and an exponential distribution.

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